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Plane waves, integrable quantum systems, Green functions and the groups $SO(p, q)$, $p \leqslant q$

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Abstract. Plane waves on symmetric spaces (SS) $X \equiv SO(p,q)/SO(p) \otimes SO(q)$ of rank p, $p \leqslant q$, are constructed by realization of the irreducible representations (principal series) of the group $SO(p,q)$ in the space of infinitely differentiable homogeneous vector functions $F(y_i)$ on cones $[y_i, y_i] = 0$, $y_i \in Y_i$, with values in the representation space of the stability subgroups $SO(p-i, q-i)$, $i = 1, \ldots, p$. We define the cones $Y_i = \mathrm{Lim}\, X(\alpha_i, \ldots, \alpha_p)$, $\alpha_i \to \infty$, corresponding to the SS X related with Cartan involutive automorphism $\sigma(g) = IgI$, $g \in SO(p,q)$, where $I = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ is the metric tensor of the pseudo-Euclidean space $R^{p,q}$. Calculating Harish–Chandra c-functions the orthogonality, completeness conditions and addition theorems for plane waves are derived. The integrable n-body quantum systems related to groups $SO(p,q)$ are considered. The explicit expressions for the Green functions in the case SS X of rank $p = 1$ and the integral representation in the general case are given.

1. Introduction

The plane waves on the Lobachevski space $SO(1,3)/SO(3)$ at first were considered and used for Fourier analysis of the representations of the Lorentz group associated with this space by Shapiro [1]. The connection of this approach with the horisphere transformation (with the integral geometry) in the space $SO(1,q)/SO(q)$ was presented by Gelfand *et al* [2] in detail. Plane waves were used by Perelomov [3] in another aspect, which was the construction of the coherent states.

We consider the spaces related with a Cartan involutive automorphism of the group $SO(p,q)$, $p \leqslant q$, namely the symmetric Riemannian and pseudo-Riemannian spaces with rank equal to $p = 1, 2, \ldots$, $X \equiv SO(p,q)/SO(p) \times SO(q)$ and $Z \equiv SO(p,q)/SO(p-1,1) \times SO(1,q-1)$, respectively.

Our aim in this paper is to generalize the Wigner theory to the case of the groups $SO(p,q)$ in order to construct the plane waves on SS X and Z which are eigenfunctions of the Laplace operators on SS X and Z. We begin with consideration of the SS of rank 1: $SO(p,q)/SO(p-1,q)$, $SO(p,q)/SO(p,q-1)$, $p \leqslant q$. The plane waves on these SS are realized by construction of the maximal degenerate irreducible representations of the group $SO(p,q)$ in the space of the infinitely differentiable homogeneous functions on a cone $Y : [y,y] = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2 = 0$. The case of the non-degenerate representations of the group $SO(1,q)$ are also considered. For a generalization

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of the construction of the plane waves for the case SS $X \equiv SO(p,q)/SO(p) \otimes SO(q)$, $p \geqslant 2, 3, \ldots,$ we consider the matrix realization of the SS given by Cartan and we present the formula of the irreducible representations (principal series) of the group $SO(p,q)$ in the space of infinitely differentiable homogeneous vector functions $F(y_i)$ on a cone Y_i (see section 3) with values in the representation space of the stability (little) subgroups $SO(p-i, q-i)$, $i = 1, \ldots, p$.

There exists a number of exact results about the complete systems of quantum commuting observables (quantum integrals of motion), wavefunctions, spectra and so on. The paper 'Quantum integrable systems related to Lie algebra' by Olshanetsky and Perelomov [4] presents the results obtained in these subjects from a general point of view. The dynamics of some of these systems is closely related to free motion in the SS. In the case of the Riemannian SS the distance between two points is real, but in the case of pseudo-Riemannian SS the distance is piecewise defined, it has real and imaginary parts. In other words, all geodesic curves in the former case have non-compact closures and in the latter case all geodesic curves have non-compact and compact closures. This is the reason why the quantum systems related to Riemannian SS have only continuous spectra of scattering states and the quantum systems related to pseudo-Riemannian SS have continuous and discrete spectra. For the case of the quantum dynamical systems related to pseudo-Riemannian SS, see [5]. The review 'Harmonic analysis and propagators on homogenous spaces' by Comporesi [6] presents the results on the Green function on Riemannian SS.

The paper is organized as follows. In section 2 for completeness and to fix the notation we construct the plane waves on SS of rank 1 and give the well known results on the orthogonality, completeness conditions and addition theorem for these plane waves. Here we also construct the vector plane waves on SS and give the orthogonality and completeness conditions only for the case of the Lorentz group $SO(1,3)$. In this section we give the definition of the S-matrix through the matrix elements of the intertwining operator and Harish–Chandra's c-functions. In section 3 the main results on the construction of the plane waves on SS with rank $p > 1$ are presented. Using the Cartan realization of the SS X of rank p we define the matrix realizations of the cone Y from the SS X by the asymptotic method. Here we give a realization of the irreducible representations (principal series) of the group $G = SO(p,q)$. This realization is equivalent to the induced representation method, where the Iwasawa decompositions $G = KAN$ is used. Using the analysis of the intertwining operators of the maximal degenerate representations of the group $SO(p,q)$ provided in [7], we calculate Harish–Chandra c-functions and give the orthogonality and completeness conditions for the plane waves on SS X. Section 4 is devoted to Green functions on SS. We give in appendix A the scalar products in the invariant subspaces of the maximal degenerate irreducible representations of the group $SO(p,q)$ and in appendix B some formulae which have been used in calculations of the Green functions.

2. Plane waves on SS of rank 1 of the group $SO(p,q)$

In this section we construct plane waves on SS defined by quadratic forms in the space $R^{p,q}$.

2.1. Plane waves on the two-sheeted hyperboloid X:$[x,x] = x_0^2 - x_1^2 - \cdots - x_q^2 = 1,\ x_0 > 0$

Firstly, we construct the plane waves for the case of the maximal degenerate representations of the group $SO(1,q)$ for simplicity. The irreducible representations of the group $SO(1,q)$

can be constructed in the space D_σ of infinitely differentiable homogeneous functions $F(y)$ with homogeneity degree σ on a cone Y:$[y, y] = y_0^2 - y_1^2 - \cdots - y_q^2 = 0 \qquad y_0 > 0$ [8]:

$$F(ay) = a^\sigma F(y) \qquad a > 0, y \in Y. \tag{2.1}$$

The representation

$$T_\sigma(g)F(y) = F(yg) \qquad g \in SO(1, q) \tag{2.2}$$

can be realized in the space of infinitely differentiable functions on intersections of the cone, for example on the sphere $S^{q-1} = SO(q)/SO(q-1)$:

$$T_\sigma(g)f(s) = (sg)_0^\sigma f(\overline{sg}) \qquad g \in SO(1, q) \tag{2.3}$$

where $s = y|_{y_0=1} = (1, \xi),\ y = \mathrm{e}^a s,\ [\xi, \xi] = 1$.

The compact realization of the representation given by equation (2.3) is compulsory, because the stabilizer subgroup of the hyperboloid X is compact. From $y' = yg$ we have

$$(sg)_0 = \mathrm{e}^{\alpha'-\alpha} \qquad (\overline{sg}) = (sg)/(sg)_0 = (1, \xi_g). \tag{2.4}$$

The unitary representation $T_\sigma(g)$ with respect to the scalar product

$$(f_1, f_2)_\sigma = \int \overline{f_1(s)} f_2(s)\,\mathrm{d}s \tag{2.5}$$

is defined by $\sigma = -(q-1)/2 + \mathrm{i}\rho,\ \rho \in [0, \infty)$ (principal series). Here $\mathrm{d}s$ is the invariant volume on the sphere S^{q-1}. Indeed, from the relations $\mathrm{d}y = \mathrm{d}y' : \mathrm{e}^{(q-1)\alpha}\,\mathrm{d}\alpha\,\mathrm{d}s = \mathrm{e}^{(q-1)\alpha'}\,\mathrm{d}\alpha'\,\mathrm{d}s'$ and $\mathrm{e}^{\alpha'} = (sg)_0 \mathrm{e}^\alpha$ we have

$$\mathrm{d}s = (sg)_0^{q-1}\,\mathrm{d}\overline{sg} = (sg)_0^{q-1}\,\mathrm{d}s'. \tag{2.6}$$

It follows that the invariance condition of the scalar product (f_1, f_2) under representation (2.3) is $\sigma + \bar{\sigma} + q - 1 = 0$.

The representation $T_\sigma(g)$ is also invariant with respect to the scalar product

$$(f_1, f_2)_\sigma = \int_{S^{q-1}} \int_{S^{q-1}} [s^{(1)}, s^{(2)}]^{-\sigma-q+1} f_1(s^{(1)}) f_2(s^{(2)})\,\mathrm{d}s^{(1)}\,\mathrm{d}s^{(2)} \tag{2.7}$$

where $-q + 1 < \sigma < 0$ (complementary series) or σ is a non-negative integer $\sigma = 1 = 0, 1, 2\ldots$ (discrete series). The representations $T_\sigma(g)$ and $T_{-\sigma-q+1}(g),\ g \in SO(1, q)$, are equivalent when σ is not an integer and partially equivalent when σ is an integer. The orthonormal basis functions on the sphere S^{q-1} are the matrix elements $D^{\{l\}}(k), k \in SO(q)$, of the representations of the group $SO(q)$ with conditions

$$D^{\{l\}}(rk) = D^{\{l\}}(k) \qquad k \in SO(q), r \in SO(q-1). \tag{2.8}$$

Here $\{l\}$ is the set of numbers which define the irreducible representations of the group $SO(q)$. Note that this realization of the representation coincides with the induced representation method where the Iwasawa decomposition $G = NAK$ of the group $SO(1, q)$ is used. Indeed the stability subgroup of the fixed point $\mathring{y} = (1, 0, \ldots, 0, 1)$ of the cone $[y, y] = 0,\ y_0 > 0$ is NM with $M = SO(q-1)$ being the centralizer of A in K. Hence the cone Y is defined as the factor space $Y = G/NM$. This definition of the cone Y is given for an abstract group G.

The restriction of the representation $T_\sigma(g)$ on the subgroup $SO(q)$ has the form

$$T_\sigma(k_0)D^{\{l\}}(k) = D^{\{l\}}(kk_0) \qquad k_0 \in SO(q). \tag{2.9}$$

Hence the representation space D_σ has the invariant vector $|0_k\rangle$ (defined by $\{l\} = 0$) with respect to the representation $T_\sigma(k)$, $k \in SO(q)$. The spherical functions of the representation $T_\sigma(g)$ are defined by the formula

$$t^\sigma_{0_k;\{l\}}(g) = \langle 0_k|T_\sigma(g)D^{\{l\}}(h(\xi))\rangle = \int_{S^{q-1}} (sg)^\sigma_0 D^{\{l\}}(h(\xi_g))\,\mathrm{d}\xi \tag{2.10}$$

where $k = rh(\xi)$, $kg = k_g = r_g h(\xi_g)$ and $\mathrm{d}\xi$ is the invariant volume on a sphere S^{q-1}.

Using the decomposition $g = kg_x$ of the elements $g \in SO(1, q)$ related to the hyperboloid X:$[x, x] = x_0^2 - x_1^2 - \cdots - x_q^2 = 1$, $x_0 > 0$: $x = \mathring{x} g_x$, $\mathring{x}k = \mathring{x}$, $\mathring{x} = (1, 0, \ldots, 0)$ we also have

$$t^\sigma_{0_k;\{l\}}(g_x) = \langle T_{-q-1-\sigma}(g_x^{-1})1|D^{\{l\}}(h(\xi))\rangle = \int (sg_x^{-1})^\sigma_0 D^{\{l\}}(h(\xi))\,\mathrm{d}\xi. \tag{2.11}$$

Since $(sg_x^{-1})_0 = [\mathring{x}, sg_x^{-1}] = [\mathring{x}g_x, s] = [x, s]$, we obtain

$$t^\sigma_{0_k;\{l\}}(g_x) = \int [x, s]^\sigma D^{\{l\}}(h(\xi))\,\mathrm{d}\xi. \tag{2.12}$$

The matrix elements $t^\sigma_{0_k;\{l\}}(g_x)$ define the basis functions on the hyperboloid $[x, x] = 1$, $x_0 > 0$. The functions $[x, s]^\sigma$ are called plane waves on this hyperboloid. The Laplace–Beltrami operator Δ_{LB} on the hyperboloid is the Casimir operator of the quasi-regular representation:

$$T(g)F(x) = F(xg). \tag{2.13}$$

Therefore the matrix elements $t^\sigma_{0_k;\{l\}}(g_x)$ satisfy the equation

$$\Delta_{\mathrm{LB}} t^\sigma_{0_k;\{l\}}(g_x) = -\sigma(\sigma + q - 1)t^\sigma_{0_k;\{l\}}(g_x). \tag{2.14}$$

Hence we have

$$\Delta_{\mathrm{LB}}[x, s]^\sigma = -\sigma(\sigma + q - 1)[x, s]^\sigma. \tag{2.15}$$

The plane waves $[x, s]^\sigma$ present the simplest realization of the solution of the eigenfunction problem for the Laplace–Beltrami operator on the hyperboloid. To define the orthogonality and completeness conditions of the functions $[x, s]^\sigma$ we consider zonal spherical functions

$$\begin{aligned} t^\sigma_{0_k;0_k}(g) = t^\sigma_{0_k;0_k}(a(\alpha)) &= \int_{S^{q-1}} [x, s]^\sigma\,\mathrm{d}\xi \\ &= \frac{\Gamma(q/2)}{\Gamma((q-1)/2)} \int_0^\pi (\cosh\alpha - \sinh\alpha\cos\theta)^\sigma \sin^{q-2}\theta\,\mathrm{d}\theta. \end{aligned} \tag{2.16}$$

Here we used the Cartan decomposition $g = kak'$, $g \in SO(1, q)$; in other words, the spherical coordinate system $x_0 = \cosh\alpha$, $\bar{x} = \sinh\alpha\bar{\eta}$, $[\bar{\eta}, \bar{\eta}] = 1$ on hyperboloid X, and the invariance of the integration over the sphere S^{q-1}. The zonal spherical functions are eigenfunctions of the radial part of the Laplace–Beltrami operator:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} + (q-1)\coth\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}\right) t^\sigma_{0_k;0_k}(\alpha) = \sigma(\sigma + q - 1)t^\sigma_{0_k;0_k}(\alpha). \tag{2.17}$$

The substitution $t^\sigma_{0_k;0_k}(\alpha) = (\sinh\alpha)^{(q-1)/2}\psi(\alpha)$ reduces equation (2.16) to the Schrödinger equation with potential

$$V = [(q/2 - 1)^2 - 1/4]\sinh^{-2}\alpha. \tag{2.18}$$

Therefore quantum systems related with the Laplace–Beltrami operator on the hyperboloid $[x, x] = 1$, $x_0 > 0$, have only scattering states. Hence, plane waves $[x, s]^\sigma$, $\sigma =$

$-(q-1)/2+\mathrm{i}\rho$, $0 \leqslant \rho < \infty$, are complete, orthogonal basis functions on hyperboloids with a following completeness and orthogonality conditions:

$$(2\pi)^{-q}\int_0^\infty \frac{\mathrm{d}\rho}{|c(\rho)|^2}\int_{S^{q-1}} \mathrm{d}\xi [x,s]^\sigma [x',s]^{\bar\sigma} = \delta(x-x')$$
$$(2\pi)^{-q}\int \mathrm{d}x [x,s]^\sigma [x,s']^{\bar\sigma} = |c(\rho)|^2\delta(\rho-\rho')\delta(s-s'). \qquad (2.19)$$

Here dx is an invariant element on the hyperboloid, $c(\sigma)$ is the Harish–Chandra c-function which is defined by an asymptotic formula. From equation (2.16) we have

$$d^\sigma_{0_k;0_k}(\alpha) \underset{\alpha\to\infty}{\approx} c(\sigma)\mathrm{e}^{((q-1)/2\mathrm{i}\rho)\alpha} + \overline{c(\sigma)}\mathrm{e}^{((q-1)/2\mathrm{i}\rho)\alpha}$$

where

$$c(\sigma) = \frac{\Gamma(q/2)}{\sqrt{\pi}\,\Gamma(q/2-1/2)}\int_0^\pi (1-\cos\theta)^\sigma \sin^{q-2}\theta\,\mathrm{d}\theta = \frac{\Gamma(q/2)\Gamma(-\sigma-(q-1)/2)}{\sqrt{\pi}\,2^{\sigma+1}\Gamma(-\sigma)}. \qquad (2.20)$$

The asymptotic expression of the plane wave $[x,s]^{\sigma-q-1}$ when $\alpha\to\infty$ defines the kernel of the intertwining operator

$$AT_\sigma(g) = T_{-q+1-\sigma}(g)A \qquad (2.21)$$

and

$$AD^{\{l\}}(h(\xi)) = \int_{S^{q-1}} [s,s']^{-q+1-\sigma} D^{\{l\}}(h(\xi'))\,\mathrm{d}\xi' = A_l(\sigma)D^{\{l\}}(h(\xi)). \qquad (2.22)$$

The S-matrix of the quantum system with potential

$$V = [l(l+q-2)+(q/2-1)^2-1/4]\sinh^{-2}\alpha \qquad (2.23)$$

and energy $E=\rho^2>0$ is given by the formula

$$S_l(E) = \frac{A_l(\sigma)}{\overline{A_l(\sigma)}} \qquad \sigma = -\frac{q-1}{2}+\mathrm{i}\sqrt{E}. \qquad (2.24)$$

It is clear that for the states with $l=0$ we have

$$S_0(E) = \frac{c(\sigma)}{\overline{c(\sigma)}}. \qquad (2.25)$$

The addition theorem for the plane waves follows from the group relations

$$t^\sigma_{0_k;0_k}(g_{x_1}g_{x_2}^{-1}) = \sum_{\{l\}} t^\sigma_{0_k;\{l\}}(g_{x_1})t^\sigma_{\{l\};0_k}(g_{x_2}^{-1}) = \sum_{\{l\}} t^\sigma_{0_k;\{l\}}(g_{x_1})\overline{t^\sigma_{0_k;\{l\}}(g_{x_2})}. \qquad (2.26)$$

Using the completeness of the basic functions on the sphere S^{q-1} we get the addition theorem for plane waves on the hyperboloid:

$$(2\pi)^{-q}\int_{S^{q-1}} \mathrm{d}s[x_1,s]^\sigma[x_2,s]^{\bar\sigma} = \int_{S^{q-1}} \mathrm{d}s[\overset{\circ}{x} g_{x_1}g_{x_2}^{-1}, s]^\sigma. \qquad (2.27)$$

The equation $[x,y]=\text{constant}(y=\mathrm{e}^\alpha s)$ defines the horisphere in the Lobachevsky space $SO(1,q)/SO(q)$, i.e. a sphere with origin at infinity which is on a cone Y. Hence, this horisphere is an analogue to the Euclidean plane. Also, if we introduce the dimensional value c, the light velocity, then the Lobachevsky space reduces to the Euclidean one and the Laplace–Beltrami operator to the square of the momentum operator, and the plane waves $[x,s]^{(q-1)/2+\mathrm{i}\rho}$ reduce to the ordinary plane waves $\mathrm{e}^{\mathrm{i}pr}$ (where p is the momentum and r the coordinate of the q-dimensional Euclidean space), when the velocity c tends to infinity.

Now we consider non-degenerate representations of the group $SO(1, q)$ to construct vector plane waves on SS X. The non-degenerate irreducible representations of the group $SO(1, q)$ are constructed in the space of infinitely differentiable vector functions $F(y)$ on a cone $[y, y] = 0$, $y_0 > 0$ with values in the representation space of the stability (little) subgroup $SO(q-1)$ of the point $\overset{\circ}{y} = (1, 0, \ldots, 0, 1)$ on the cone. This representation can be constructed in the space of infinitely differentiable vector functions $f(s)$ on a sphere S^{q-1}. The orthonormal basis functions on the sphere S^{q-1} are the matrix elements $D^{\{l\}}(k)$, $k \in SO(q)$, of the representations of the group $SO(q)$ with covariant conditions

$$D^{\{l\}}(rk) = D^{\{l'\}}(r) D^{\{l\}}(k) \qquad k \in SO(q), r \in SO(q-1). \tag{2.28}$$

Here $D^{\{l'\}}(r)$ are the matrix elements of the representations of the group $SO(q-1)$.

The representation formula has the form

$$T_{\chi(\sigma,\{l\})}(g) D^{\{l\}}(k) = (sg)_0^{\sigma} D^{\{l\}}(k_g) \qquad g \in SO(1, q), k \in SO(q). \tag{2.29}$$

For the restriction on the maximal compact subgroup $SO(q)$ we have

$$T_{\chi(\sigma,\{l\})}(k_0) D^{\{l\}}(k) = D^{\{l\}}(kk_0) \qquad k, k_0 \in SO(q). \tag{2.30}$$

From equation (2.23) we obtain

$$t^{\chi}_{\{l'\};\{l\}}(g) = \int (sg)_0^{\sigma} \overline{D^{\{l'\}}(k)} D^{\{l\}}(k_g)\, \mathrm{d}k. \tag{2.31}$$

Hence

$$t^{\chi}_{\{l'\};\{l\}}(g_x) = \int [x, s]^{\sigma} \overline{D^{\{l'\}}(k_{g_x^{-1}})} D^{\{l\}}(k)\, \mathrm{d}k. \tag{2.32}$$

So, plane waves on the hyperboloid for the case of the non-degenerate representations have the form

$$[x, s]^{\sigma} \overline{D^{\{l'\}}(k_{g_x^{-1}})}. \tag{2.33}$$

We demonstrate the calculation of the $c[\chi(\sigma, \{l\})]$ functions for the case of the Lorentz group $SO(1, 3)$. In this case representations $T_{\chi}(g)$, $g \in SO(1, 3)$, $\chi = (\sigma, \nu)$, are defined by $\sigma = -1 + \mathrm{i}\rho$, $\rho \in [0, \infty)$, and by the representation $T_{\nu}(k)$, $k \in SO(2)$, $\nu = 0, \pm 1/2, \pm 1, \ldots$ of the stability subgroup $SO(2)$. The basis functions on the sphere S^2 are the orthonormal Wigner D-functions:

$$D^{(J)}_{\nu\lambda}(0, \theta, \varphi) = d^{(J)}_{\nu\lambda}(\theta) \mathrm{e}^{\mathrm{i}\lambda\varphi}. \tag{2.34}$$

The integral representations of the matrix element of the representation $T_{\chi}(a(\alpha))$, $a \in SO(1, 1)$ have the form

$$t^{\chi}_{s\lambda;J}(a(\alpha)) = (2\pi)^2 \int_0^{\pi} (\cosh\alpha - \sinh\alpha\cos\theta)^{\sigma} d^{J}_{\nu\lambda}(\theta_a) \overline{d^{(s)}_{\nu\lambda}(\theta)} \sin\theta\, \mathrm{d}\theta. \tag{2.35}$$

Here

$$\cos\theta_a = \frac{\sinh\alpha + \cosh\alpha\cos\theta}{\cosh\alpha + \sinh\alpha\cos\theta} \qquad \varphi_a = \varphi.$$

It follows that $\cos\theta_a \to 1$ when $\alpha \to \infty$. Because $d^{\sigma}_{\nu\lambda}(\theta = 0) = \delta_{\nu\lambda}$ from equation (2.35) we obtain

$$t^{\chi}_{s\lambda;J}(\alpha) \underset{\alpha\to\infty}{=} (2\pi)^2 \mathrm{e}^{\sigma\alpha} \delta_{\nu\lambda} \int_0^{\pi} (1 - \cos\theta)^{\sigma} \overline{d^{(s)}_{\nu\lambda}(\theta)} \sin\theta\, \mathrm{d}\theta. \tag{2.36}$$

The $|c(\chi)|^2$ functions depend only on the weight of the representation. Putting $s = \nu$ in equation (2.34) we have

$$|c(\chi)|^2 = \left(\left(\frac{\rho}{2}\right)^2 + \nu^2\right)^{-1} \tag{2.37}$$

which is the Plancherel measure in the representation space of the Lorentz group. So the orthogonality and completeness conditions for the vector plane waves

$$[x, s]^\sigma \overline{D^{(s)}_{\nu\lambda}(k_{g_x^{-1}})} \tag{2.38}$$

on the hyperboloid $[x, x] = 1$, $x_0 > 0$, have the form

$$\frac{1}{(2\pi)^4} \sum_{\lambda=-s}^{s} \int_{[x,x]=1} [x, s]^\sigma \overline{D^{(s)}_{\nu\lambda}(k_{g_x^{-1}})} [x, s']^{\overline{\sigma'}} D^{(s')}_{\nu'\lambda}(k_{g_x^{-1}})\, \mathrm{d}x = \frac{\delta_{\nu\nu'}\delta(\rho - \rho')\delta(s - s')}{(\rho/2)^2 + \nu^2} \tag{2.39}$$

and

$$\frac{1}{(2\pi)^4} \sum_{\nu=-s}^{s} \int_0^\infty \int_{S^2} [x, s]^\sigma D^{(s)}_{\nu\lambda}(k_{g_x^{-1}}) [x', s]^{\bar{\sigma}} \overline{D^{(s)}_{\nu\lambda'}(k_{g_x^{-1}})} ((\rho/2)^2 + \nu^2)\, \mathrm{d}\rho\, \mathrm{d}s = \delta_{\lambda\lambda'}\delta(x - x').$$

The expansion of the amplitude of the one-particle helicity state $F_{s\lambda}(p)$ with mass $m^2 = E^2 - (\boldsymbol{p})^2$ and helicity λ by this vector plane wave have been obtained in [9]. The generalization of the Gelfand–Graev integral transformation in Lobachevsky space for a non-degenerate representation was obtained in [10].

2.2. Plane waves on the one-sheeted hyperboloid Z:$[z, z] = z_0^2 - z_1^2 - \cdots - z_q^2 = -1$

The plane waves on the hyperboloid Z:$[z, z] = -1$ are defined by the method of analytical continuation from the plane waves on hyperboloid X [11]. Noting that the transition $x \to z \in Z$ can be obtained by one $\alpha \to \alpha \pm \mathrm{i}\pi/2$ in the spherical coordinate system, we have

$$[x, s]^\sigma \to ([z, s] \pm \mathrm{i}0)^\sigma \cdot \mathrm{e}^{\pm\mathrm{i}\pi\sigma/2}. \tag{2.40}$$

Correspondingly the potential (2.18) is replaced by

$$V = -[(q/2 - 1)^2 - 1/4] \cosh^{-2}\alpha. \tag{2.41}$$

Therefore quantum systems related with the Laplace–Beltrami operator on the hyperboloid Z have scattering and bound states. It is convenient, instead of having two independent homogenous generalized functions (distributions) $(t \pm \mathrm{i}o)^\sigma$, to consider the generalized functions $|t|^\sigma \operatorname{sign}^\varepsilon t$, $\varepsilon = 0, 1$, using the relations

$$2\cos\frac{\pi\sigma}{2}|t|^\sigma = \mathrm{e}^{-\mathrm{i}\pi\sigma/2}(t + \mathrm{i}0)^\sigma + \mathrm{e}^{\mathrm{i}\pi\sigma/2}(t - \mathrm{i}0)^\sigma \tag{2.42}$$

and

$$2\mathrm{i}\sin\frac{\pi\sigma}{2}|t|^\sigma \operatorname{sign} t = -\mathrm{e}^{-\mathrm{i}\pi\sigma/2}(t + \mathrm{i}0)^\sigma + \mathrm{e}^{\mathrm{i}\pi\sigma/2}(t - \mathrm{i}0)^\sigma. \tag{2.43}$$

Thus we define the plane waves on the hyperboloid Z as

$$|[z, s]|^\sigma_\varepsilon \equiv \cos\pi(\sigma + \varepsilon)/2 |[z, s]|\sigma \overset{\varepsilon}{\operatorname{sign}}[z, s] \qquad \varepsilon = 0, 1 \tag{2.44}$$

where $\sigma = -(q - 1)/2 + \mathrm{i}\rho$, $0 < \rho < \infty$, for scattering states and

$$|[z, s]|^{-l-q-+1}_\varepsilon \equiv \operatorname*{Lim}_{\sigma\to -l-q+1} \cos\pi(\sigma + \varepsilon)/2 |]z, s]|^\sigma \overset{\varepsilon}{\operatorname{sign}}[z, s] \tag{2.45}$$

for discrete states.

Note that the generalized function $|t|^\sigma$ of one variable has a simple pole at $\sigma = -2m-1$ with residue

$$\mathop{\mathrm{Re}\,s}_{\sigma=-2m-1}|t|^\sigma = \frac{[2\delta^{2m}(t)]}{\Gamma(2m+1)}. \tag{2.46}$$

However, if $\sigma = -2m$ the generalized function $|t|^\sigma$ is regular at $\sigma = -2m$ and takes the value t^{-2m} at this point. Hence we have

$$\begin{aligned} &\mathop{\mathrm{Lim}}_{\sigma\to-2m-1}\cos\frac{\pi\sigma}{2}|[z,s]|^\sigma = (-1)^m\frac{\pi}{\Gamma(2m+1)}\delta^{2m}([z,s]) \\ &\mathop{\mathrm{Lim}}_{\sigma\to-2m-1}2\mathrm{i}\sin\frac{\pi\sigma}{2}|[z,s]|^\sigma\,\mathrm{sign}[z,s] = 2\mathrm{i}(-1)^{m+1}[z,s]^{-2m-1} \\ &\mathop{\mathrm{Lim}}_{\sigma\to-2m}\cos\frac{\pi\sigma}{2}|[z,s]|^\sigma = (-1)^m[z,s]^{-2m}. \end{aligned} \tag{2.47}$$

The orthogonality condition for plane waves on the hyperboloid Z has the form

$$(2\pi)^{-q}\int \mathrm{d}z[z,s]_\varepsilon^\sigma[z,s']_{\varepsilon'}^{-\sigma'-q+1} = c(\sigma)c(-\sigma-q+1)\delta(\sigma-\sigma')\delta(s-s')\delta_{\varepsilon\varepsilon'}. \tag{2.48}$$

For the scattering states, Harish–Chandra's $c(\sigma)$-function is the same as in equation (2.20), but for the discrete states relation (2.48) is understood as a limit case when $\sigma \to -l-q-1$. The completeness condition has the form

$$\begin{aligned} &\sum_{g=0,1}\int_0^\infty \mathrm{d}\rho/|c(\rho)|^2\int_{s^{q-1}}\mathrm{d}s[z,s]_\varepsilon^{(-(q-1)/2+(\mathrm{i}\rho)}[z',s]_\varepsilon^{(-(q-1)/2+\mathrm{i}\rho)} \\ &\qquad + \sum_{g=0,1}\int_\gamma \mathrm{d}\sigma/|c(\sigma)c(-q+1-\sigma)|\int_{s^{q-1}}\mathrm{d}s[z,s]_\varepsilon^\sigma[z',s]_\varepsilon^{-q+1-\sigma} \\ &\quad = (2\pi)^q\delta(z-z') \end{aligned} \tag{2.49}$$

where the contour γ encloses the simple poles of the function $1/c(\sigma)$

2.3. Plane waves on the hyperboloid $Z_\pm:[z,z] = z_1^2+\cdots+z_p^2-z_{p+1}^2-\cdots-z_{p+q}^2 = \pm 1$

Now we construct the plane waves on the hyperboloids $Z_\pm:(z_1^2+\cdots+z_p^2-z_{p+1}^2-\cdots-z_{p+q}^2 = \pm 1, z = (z_1,\ldots,z_p) \in Z_\pm$, of the group $SO(p,q)$, $1 < p < q$. In order to do this we consider the maximal degenerate representation of the group $SO(p,q)$ in the space D_χ, $\chi = (\sigma,\varepsilon)$, of infinitely differentiable homogenous functions $F(y)$ with homogeneity degree σ and parity ε on a cone Y $(y_1^2+\cdots+y_p^2-y_{p+1}^2-\cdots-y_{p+q}^2 = 0, y = (y_1,\ldots,y_p) \in Y)$

$$F(ay) = |a|^\sigma\overset{\varepsilon}{\mathrm{sign}}\,aF(y) \qquad \varepsilon = 0,1.$$

The representation

$$T_\chi(g)F(y) = F(yg) \qquad \chi = (\sigma,\varepsilon) \qquad g \in SO(p,q) \tag{2.50}$$

can be realized in the space of infinitely differentiable functions on intersections of the cone Y with the plane $y_1 = 1{:}y = dp$, $p = y|_{y=1}$, $p = (1,\eta)$, $\eta = (\eta_1,\ldots,\eta_{p-1}\eta_p,\ldots,\eta_{p+q-1})$, $[\eta,\eta] = -1$, $\eta \in Z$. From equation (2.50) we have

$$T_\chi(g)f(p) = |(pg)_1|^\sigma\overset{\varepsilon}{\mathrm{sign}}(pg)_1 f(\overline{pg}) \qquad g \in SO(p,q). \tag{2.51}$$

Here we used $d'/d = (pg)_1$, $\overline{pg} = (pg)/(pg_1)$, which follow from $y' = yg$. The expression $(pg)_1$ can be rewritten in the form

$$(pg)_1 = \overset{\circ}{z}, pg]$$

where $\overset{\circ}{z} = (1, 0, \ldots 0; 0 \ldots 0)$ is a fixed point of the hyperboloid Z_+. Then for $g \equiv g_{z_0}^{-1}$ where g_z is defined by relation $z = \overset{\circ}{z} g_z$ we have

$$(pg_z^{-1})_1 = [z, p].$$

The non-compact realization of the representation given by equation (2.51) is compulsory, because the stabilizer subgroup of the hyperboloid Z_+ is non-compact. The complete basis functions on the $(p + q - 1)$-dimensional hyperboloid Z_+ are the matrix elements $t^\chi(hg)$ of the representation of the group $SO(p, q)$ with conditions

$$t^\chi(hg) = t^\chi(g) \qquad h \in H \qquad H \equiv SO(p - 1, q) \qquad g \in SO(p, q). \tag{2.52}$$

Let 1 be an H invariant dual vector of the representation space, then we have

$$t^\chi_{0_H;\chi'}(gz) = \langle T_{-p-q+2-\sigma,\varepsilon}(g_z^{-1}) 1 / D^{\chi'}(h(\eta))\rangle = \int |[z, p]|^{-p-q+2-\sigma} \overset{\varepsilon}{\mathrm{sign}}[z, p] D^{\chi'}(g_\eta)\, \mathrm{d}\eta \tag{2.53}$$

where $\eta = \overset{\circ}{\eta} g_\eta$, $\overset{\circ}{\eta} = (0, \ldots, 0, 1)$, is the vector of the $(p + q - 2)$-dimensional hyperboloid Z_-, $\mathrm{d}\eta$ is an invariant volume element on Z_-. Hence, to evaluate the basis functions on the $(p + q - 1)$-dimensional hyperboloid Z_+, we need the basis functions on the $(p + q - 2)$-dimensional hyperboloid Z_-. It can be done step by step. It is convenient to evaluate the matrix elements of the representations $T_\chi(g)$, $g \in SO(p, q)$, in the mixed basis. For this we consider the realization of the representation (2.44) in the space of the functions on a compact section $r^2 = y_1^2 + \cdots + y_p^2 = y_{p+1}^2 + \cdots + y_{p+q}^2 = 1$ of the cone Y. That is $y = rs$, $r > 0$, $s = (\xi^{(p)}, \xi^{(q)})$, $\xi^{(p)} \in S^{p-1}$, $\xi^{(q)} \in S^{q-1}$, $s \in S^{p-1} \times S^{q-1}$. Then we have

$$T_\chi(g)\varphi(s) = (r_g/r)^\sigma \varphi(\overline{sg}),\ \chi = (\sigma, \varepsilon) \qquad \text{where } \varphi(-s) = (-1)^\varepsilon \varphi(s). \tag{2.54}$$

Using the relations

$$f(p) = \left(\frac{r}{|d|}\right)^\sigma \overset{\varepsilon}{\mathrm{sign}}\, d\psi(s) \qquad [x, p] = \frac{r}{d}[x, s] \qquad \mathrm{d}\eta = \left(\frac{r}{d}\right)^{p+q-2} \mathrm{d}s \tag{2.55}$$

and replacing the basis functions on hyperboloid Z_- by the basis functions on sections $S^{p-1} \times S^{q-1}$ in equation (2.53) we have

$$t^x_{0_H;\{l_p\};\{l_q\}}(g_z) = \int_{S^{p-1}} \int_{S^{q-1}} |[z, s]|^{-p-q+2-\sigma} \overset{\varepsilon}{\sin}[z, s] \times D^{\{l_p\}}(\xi^{(p)}) D^{\{l_q\}}(\xi^{(q)})\, \mathrm{d}\xi^{(p)}\, \mathrm{d}\xi^{(q)}. \tag{2.56}$$

The matrix elements $t^x_{0_H;\{l_p\};\{l_q\}}(g_z)$ are the basis functions on the $(p + q - 1)$-dimensional hyperboloid Z_+ given in the bispherical coordinate system and satisfy the equation

$$\Delta_{\mathrm{LB}} t^x_{0_H;\{l_p\};\{l_q\}}(g_z) = -\sigma(\sigma + p + q - 2) t^x_{0_H;\{l_p\};\{l_q\}}(g_z). \tag{2.57}$$

Hence for the plane waves $[z, s]_{\sigma,\varepsilon} \equiv |[z, s]|^\sigma \mathrm{sign}^\varepsilon[z, s]$ we have

$$\Delta_{\mathrm{LB}}[z, s]_{\sigma,\varepsilon} = -\sigma(\sigma + p + q - 2)[z, s]_{\sigma,\varepsilon}.$$

To define the orthogonality and completeness condition of the plane waves $[z, s]_{\sigma,\varepsilon}$ we consider the 'zonal' spherical function

$$t^x_{0_H;0_k}(g) \equiv t^x_{0_H;0_k}(\alpha) = \int_{S^{p-1}S^{q-1}} [z, s]_{\sigma,0}\, \mathrm{d}s = \frac{\Gamma(q/2)\Gamma(p/2)}{\Gamma(q - 1/2)\Gamma(p - 1/2)}$$
$$\times \int_0^\pi \int_0^\pi |\cosh\alpha \cos\omega - \sinh\alpha \cos\theta|^\sigma x \sin^{p-2}\omega \sin^{q-2}\theta\, \mathrm{d}\omega\, \mathrm{d}\theta. \tag{2.58}$$

Here we used the bispherical coordinate system on the hyperboloid

$$Z_+ \colon z = (\mathrm{ch}\alpha\, \xi^{(p)}, \mathrm{sh}\alpha\, \xi^{(q)}) \qquad \xi^{(p)} \in S^{p-1}, \xi^{(q)} \in S^{q-1}$$

and the invariance of the integration over the spheres S^{p-1} and S^{q-1}. The 'zonal' spherical functions are eigenfunctions of the radial part of the Laplace–Beltrami operator given by

$$\left[\frac{1}{\sinh^{q-1}\alpha\cosh^{p-1}\alpha}\frac{\mathrm{d}}{\mathrm{d}\alpha}\sinh^{q-1}\alpha\cosh^{p-1}\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}\right]t^{(\sigma,0)}_{0_k;0_H}(\alpha)=\sigma(\sigma+p+q-2)t^{(\sigma,0)}_{0_k;0_H}(\alpha). \tag{2.59}$$

The substitution

$$t^{(\sigma,0)}_{0_k;0_H}(\alpha)=(\sinh\alpha)^{-(q-1)/2}(\cosh\alpha)^{-(p-1)/2}\psi^{\sigma}(\alpha) \tag{2.60}$$

reduces equation (2.59) to the Schrödinger equation with potential

$$V=\frac{(q-1)(q-3)/4}{\sinh^2\alpha}-\frac{(p-1)(p-3)/4}{\cosh^2\alpha}. \tag{2.61}$$

The scattering and bound states of this quantum system are defined by the principal (when $\sigma=-(p+q-2)/2+\mathrm{i}\rho, 0<\rho<\infty$) and discrete series (when $\sigma=n, n=0,1,2\ldots$) of the irreducible unitary representation of the group $SO(p,q)$. In the study of the irreducible representations of the group $SO(p,q)$ it is convenient to work with the compact realization of the representation (2.50) as was done in [7, 12, 13].

It follows from equation (2.58) that Harish–Chandra's c-function is given by the formula

$$C_{p,q}(\sigma)=\frac{\Gamma(q/2)\Gamma(p/2)}{\Gamma(q-1/2)\Gamma(p-1/2)}\int_0^{\pi}\int_0^{\pi}|\cos\omega-\cos\theta|^{\sigma}\sin^{p-2}\omega\sin^{q-2}\theta\,\mathrm{d}\omega\,\mathrm{d}\theta. \tag{2.62}$$

This integral was evaluated in [7] and we have (see appendix A)

$$c_{p,q}(\sigma)=\frac{\Gamma(p/2)\Gamma(q/2)\Gamma(-\sigma-(p+q-2)/2)\Gamma((-\sigma-q-p+3)/2)}{\sqrt{\pi}\Gamma(-\sigma/2)\Gamma((-\sigma-p+2)/2)\Gamma((-\sigma-q+2)/2)}. \tag{2.63}$$

The orthogonality and completeness conditions for plane waves $[z,s]_{\sigma,\varepsilon}$ on hyperboloid Z_+ are given by a formula similar to (2.48) where the c-function is replaced by the $c_{p,q}$-function.

3. Plane waves on SS of rank $p>1$ of the group $SO(p,q)$

For construction of the plane waves on SS X of rank p we use Cartan's realization of the symmetric homogeneous space, which is defined by the involutive automorphism σ: $\sigma^2=1$. For the pseudo-orthogonal group $G=SO(p,q)$ for which $gIg^t=I$, $g\in G$, $I=\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ is the metric's tensor of the pseudo-Euclidean space $R^{p,q}$, we define the involutive automorphism

$$\sigma(g)=IgI. \tag{3.1}$$

Hence the elements of the maximal compact subgroup $K=SO(p)\otimes SO(q)$ are fixed under this automorphism: $\sigma(k)=k$, $k\in K$. The homogeneous space $X=G/K$ with motion group G and stabilizer K is called symmetric. An action of the group G on SS X is defined by

$$g\circ x=x'=gx\sigma(g^{-1}) \qquad x\in X, g\in G. \tag{3.2}$$

The stabilizer K fixes the point $\overset{\circ}{x}=1$-identity, and we have a realization of the space X in the form

$$x=g_x\sigma(g_x^{-1}) \qquad x\in X. \tag{3.3}$$

The space X for the group $SO(p,q)$, $p\leqslant q$, is the SS of rank p. In the general case, for any group, the rank of the homogeneous space is equal to the number of independent invariant differential operators on this space. For the Cartan decomposition $G = KAK'$ of the group $SO(p,q)$ we have

$$x = ka^2k^{-1} \qquad k \in K, a \in A \tag{3.4}$$

where $a(\alpha', \dots, \alpha^p) = \prod_{J=1}^{p} a(\alpha^j)$ are hyperbolic rotations in planes $(x^j, x^{p+q+1-j})$, $j = 1, p, k = \mathrm{diag}(k^p k^q), k^p \in SO(p), k^q \in SO(q)$. The parameters $\alpha^1, \dots, \alpha^p$ is called Cartan coordinates on SS. The metric on the SS is induced by the metric in the pseudo-Euclidean space of the $n \otimes n (n = p + q)$ matrices:

$$[x_1, x_2] = \tfrac{1}{2}\,\mathrm{tr}(Ix_1Ix_2^t). \tag{3.5}$$

Hence the group $SO(p,q)$ defines the hyperboloid $[x,x] = n/2$ in this space. The metric matrix in the SS is given by the formula

$$g_{ij} = \frac{1}{n}\,\mathrm{tr}(I\dot{x}_{t_i}I\dot{x}_{t_j}) \tag{3.6}$$

where $\dot{x}_{t_i} = \mathrm{d}x/\mathrm{d}t$, t_i are the coordinates of the SS. The radial part of the Laplace–Beltrami operator on SS is defined by the formula [14]

$$\frac{1}{\sqrt{g}} \sum_{j=1}^{p} \frac{\partial}{\partial \alpha^j} \sqrt{g} \frac{\partial}{\partial \alpha^j} \tag{3.7}$$

where $\sqrt{g} = \sqrt{\det(g_{ij})} = \prod_{i<j}^{p} \sinh(\alpha_i - \alpha_j)\sinh(\alpha_i + \alpha_j)\sinh^{q-p}\alpha_i$. Here the Cartan coordinates $\alpha^i - \alpha^j$, $\alpha^i + \alpha^j$, α^i, $i < j$ correspond to positive restricted roots of the algebra of the group $SO(p,q)$ with multiplicity 1, 1 and $q - p$, respectively [14].

Using the action formula (3.2) of the transformation group G on SS X we define the quasi-regular representation of the group $SO(p,q)$ by the formula

$$T(g)f(x) = f(gx\sigma(g^{-1})) \qquad x \in X, g \in G. \tag{3.8}$$

For decomposition of the quasi-regular representation into irreducible components we will use the plane waves on SS X. In order to define the plane waves on (SS) X of rank p, $p \leqslant q$, we construct the irreducible representations of the group $T_{\chi_1}(g)$, $\chi_1 = (\sigma_1, \chi_2)$, $g \in SO(p,q)$, in the space D_{χ_1} of infinitely differentiable homogeneous vector functions $F(y)$ on a cone Y_1 with homogeneity degree σ_1 whose values belong to the space D_{χ_2} of the representation $T_{\chi_2}(\breve{g})$, $\breve{g} \in SO(p-1, q-1)$. We define the cone Y_1 as asymptotic to the SS X:

$$y_1 = \operatorname*{Lim}_{\alpha^1 \to \infty} x = \operatorname*{Lim}_{\alpha^1 \to \infty} ka^2k^{-1} = \mathrm{e}^{2\alpha^1} k \mathring{y}_1 k^{-1} \qquad y_1 \in Y_1, x \in X \tag{3.9}$$

where

$$\mathring{y}_1 = \begin{pmatrix} 1 & 0 & ,\dots, & 0 & 1 \\ 0 & 0 & ,\dots, & 0 & 0 \\ \ddots & \ddots & \vdots & \ddots & \ddots \\ 0 & 0 & ,\dots, & 0 & 0 \\ 1 & 0 & ,\dots, & 0 & 1 \end{pmatrix}.$$

We see that the cone (3.9) can be constructed as a homogeneous space Y_1 with the stationary point $\mathring{y}_1$, using the Iwasawa decomposition $G = NAK$:

$$y_1 = kan\mathring{y}_1 n^t a k^{-1} = \mathrm{e}^{2\alpha^1} k \mathring{y}_1 k^{-1} \qquad y_1, \mathring{y}_1 \in Y_1, k \in K, a \in A, n \in N \tag{3.10}$$

where the matrix n has the form

$$n = \begin{pmatrix} 1+\frac{1}{2}t^2 & t_1,\ldots,t_{p-1},t_p,\ldots,t_{p+q-2} & -\frac{1}{2}t^2 \\ -t_1 & & t_1 \\ \vdots & & \vdots \\ -t_{p-1} & I & t_{p-1} \\ t_p & & -t_p \\ \vdots & & \vdots \\ t_{p+q+2} & & -t_{p+q+2} \\ \frac{1}{2}t^2 & t_1,\ldots,t_{p-1},t_p,\ldots,t_{p+q-2} & 1-\frac{1}{2}t^2 \end{pmatrix}$$

where I is $(p+q-2)\times(p+q-2)$ unit matrix and $t^2 = t_1^2+\cdots+t_{p-1}^2-t_p^2-\cdots-t_{p+q-2}^2$. The representation formula can be written in the form

$$T_{\chi_1}(g)f(s_1) = \mathrm{e}^{(\alpha_g^1-\alpha^1)\sigma_1}t^{\chi_2}(\breve{g}(s_1,g))f(\overline{gs_1\sigma(g^{-1})}) \qquad g\in SO(p,q) \tag{3.11}$$

where $y_1 = \mathrm{e}^{\alpha}s_1$, $s_1 = k\mathring{y}_1k^{-1}$, $\chi_1 = (\sigma_1,\chi_2)$, $\breve{g}\in SO(p-1,q-1)$ is the stability subgroup of the point $\mathring{y}_1$; $f(s)$ is a vector function on the intersection of the cone $s_1 = k\mathring{y}_1k^{-1}$ with values in the representation space D_{χ_2} of the stability subgroup. The expressions $\mathrm{e}^{(\alpha_g^1-\alpha^1)\sigma_1}$, $\breve{g}(s_1,g)$ are defined from the relations

$$y_1' = gy_1\sigma(g^{-1}) \qquad \mathrm{e}^{-(\alpha_g^1-\alpha^1)}k'^{-1}gk = n\breve{g}(s_1,g). \tag{3.12}$$

It follows that

$$\mathrm{e}^{2(\alpha_g^1-\alpha^1)} = \tfrac{1}{2}\,\mathrm{tr}(gs_1\sigma(g^{-1})). \tag{3.13}$$

The unitary representation $T_\chi(g)$, $g\in SO(p,q)$, with respect to the scalar product

$$(f_1,f_2) = \int\langle\overline{f_1(s)}f_2(s)\rangle\,\mathrm{d}s \tag{3.14}$$

is defined by $\sigma_1 = -(p+q-2)/2+\mathrm{i}\rho_1$, $\rho_1\in[0,\infty)$ (principal series). Here $\langle|\rangle$ denotes the inner product of the vector-valued functions in the space D_{χ_2}, $\mathrm{d}s = \mathrm{d}\xi_p\,\mathrm{d}\eta_q$, $\xi_p(\eta_q) - p(q)$ are dimensional unit vectors on the spheres $S^p = SO(p)/SO(p-1)$, $S^q = SO(q)/SO(q-1)$. Indeed, using $\mathrm{d}s = \mathrm{e}^{(p+q-2)(\alpha_g-\alpha)}\,\mathrm{d}\overline{sg}$ which follows from the relation $\mathrm{d}y' = \mathrm{d}y$, $y' = gy\sigma(g^{-1})$ it is easy to verify that the invariant condition of the scalar product with respect to T_χ is $\sigma_1+\overline{\sigma_1}+p+q-2=0$. Let us put in (3.12) and (3.13) the expression

$$g = Ig_x^{-1} = Ia_x^{-1}k_x^{-1} = a_xk_x^{-1}I \qquad \sigma(g^{-1}) = Ik_xa_x. \tag{3.15}$$

Then we obtain

$$\mathrm{e}^{2(\alpha^1_{g_x^{-1}}-\alpha^1)} = \tfrac{1}{2}\,\mathrm{tr}(Ik_xa_x^2k_x^{-1}Is_1) = [x,s_1] \tag{3.16}$$

and

$$\begin{aligned}\breve{g}_x\breve{n}_x \equiv \breve{g}_1(s_1,g_x^{-1})\breve{n}_1(s_1,g_x^{-1}) &= \mathrm{e}^{-(\alpha^1_{g_x^{-1}}-\alpha^1)}k'^{-1}a_xk_x^{-1}Ik\sigma((\breve{g}_x\breve{n}_x)^{-1})\\ &\equiv \mathrm{e}^{-(\alpha^1_{g_x^{-1}}-\alpha^1)}k^{-1}Ik_xa_xk'.\end{aligned} \tag{3.17}$$

In the case of the group $SO(2,q)$ the zonal spherical functions on SS X is defined by the formula

$$t^{\chi_1}_{0_k;0_k}(Ig_x) = \langle T_\chi(Ig_x^{-1})1,1\rangle = \int[x,s_1]^{\sigma_1/2}t^{\sigma_2}_{0_k;0_k}(\breve{g}_x)\,\mathrm{d}s_1 \qquad g_x\in SO(2,q). \tag{3.18}$$

Here we have used the equivalence of the representations $\chi = (\sigma_1,\sigma_2)$, $\overline{\chi} - (\overline{\sigma_1},\overline{\sigma_2})$.

For the zonal spherical functions $t^{\sigma_2}_{0_k;0_k}$ of the stability subgroup $SO(1, q-1)$ we have a similar integral representation. Now we use the matrix realization of the cone Y_2:

$$y_2 = \breve{g}\mathring{y}_2\sigma(\breve{g}^{-1}) \qquad \breve{g} \in SO(p-1, q-1) \tag{3.19}$$

where

$$\mathring{y}_2 = \begin{pmatrix} 0 & 0 & ,\ldots, & 0 & 0 \\ 0 & 1 & ,\ldots, & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & ,\ldots, & 1 & 0 \\ 0 & 0 & ,\ldots, & 0 & 0 \end{pmatrix}.$$

Consider the transformation

$$\breve{g}_x n_x y_2 \sigma(n_x^{-1} g_x^{-1}) = y_2' + n_2 \tag{3.20}$$

where $y_2' = \breve{g}_x y_2 \sigma(\breve{g}_x^{-1})$ and the translation matrix n_2 has the form (3.10) with the matrix I equal to zero. Using (3.17) from (3.20) it follows that

$$\mathrm{e}^{2(\alpha^2_{g_x^{-1}} - \alpha^2)} = \tfrac{1}{2}[x, s_1]^{-1}\{[x, ks_2k^{-1}] - [x, s_1]\,\mathrm{tr}(n_2)\}. \tag{3.21}$$

So we have

$$t^{\chi}_{0_k;0_k}(g_x) = \int [x, s_1]^{(\sigma_1-\sigma_2)/2}\{[x, ks_2k^{-1}] - [x, s_1]\,\mathrm{tr}(n_2)\}^{\sigma_2}\,\mathrm{d}s_1\,\mathrm{d}s_2. \tag{3.22}$$

Repeating this procedure in the general case of the group $SO(p, q)$ we finally have

$$\begin{aligned} t^{\chi_1}_{0_k;0_k}(g_x) = \int & [x, s_1]^{\sum_{j=1}^{p}(-1)^{j+1}\sigma_j}\{[x, k_1s_2k_1^{-1}] - [x, s_1]\,\mathrm{tr}(n_2)\}^{\frac{1}{2}\sum_{j=2}^{2}(-1)^{j+1}\sigma_j} \\ & \times\{[x, k_{p-1}\ldots k_1 s_p k_1^{-1}\ldots k_{p-1}^{-1}] - [x, k_{p-2}\ldots k_1 s_{p-1} k_1^{-1}\ldots k_{p-2}^{-1}] \\ & \times \mathrm{tr}(n_p)\}^{\sigma_p/2}\prod_{i=1}^{p}\mathrm{d}s_i. \end{aligned} \tag{3.23}$$

Thus we obtain the plane waves on the SS X of rank p, which is the expression under the integral sign in the integral representation (3.23)

$$\begin{aligned} \exp & \left[\sum_{j=1}^{p}\sigma_j' \ln\{[x, s_j'] - [x, s_{j-1}']\,\mathrm{tr}(n_{j-1})\}\right] \\ & = [x, s_1]^{\sum_{j=1}^{p}(-1)^{j+1}\sigma_j}\{[x, k_1s_2k_1^{-1}] - [x, s_1]\,\mathrm{tr}(n_2)\}^{\frac{1}{2}\sum_{j=2}^{p}(-1)^{j+1}\sigma_j}\ldots \\ & \quad \ldots\{[x, k_{p-1}\ldots k_1 s_p k_1^{-1}\ldots k_{p-1}^{-1}] - [x, k_{p-2}\ldots k_1 s_{p-1} k_1^{-1}\ldots k_{p-2}^{-1}] \\ & \quad \times \mathrm{tr}(n_p)\}^{\sigma_p/2} \end{aligned} \tag{3.24}$$

where

$$s_j' = k_{j-1}\ldots k_1 s_j k_1^{-1}\ldots k_{j-1}^{-1} \qquad \sigma_j' = \sum_{n=j}^{p}(-1)^{n+2}\sigma_n.$$

We point out that the plane waves (3.24) are defined through the traces of the matrices. Hence one has a simpler form than plane waves based on the general theory of induced representation where the plane waves are defined through minors of the matrices $G = NAK$. The plane waves are eigenfunctions of the Laplace–Beltrami operator on SS X:

$$\Delta \exp\left[\sum_{j=1}^{p} \sigma_j' \ln\{[x, s_j'] - [x, s_{j-1}'] \operatorname{tr}(n_{j-1})\}\right] = (\sigma_1(\sigma_1 + p + q - 2) + \cdots$$
$$\cdots + \sigma_p(\sigma_p + q - p)) \exp\left[\sum_{j=1}^{p} \sigma_j' \ln\{[x, s_j'] - [x, s_{j-1}'] \operatorname{tr}(n_{j-1})\}\right]. \qquad (3.25)$$

The zonal spherical functions $\Phi(\alpha^1, \ldots, \alpha^p) = t^{\chi_1}_{0_k;0_k}(a)$ are eigenfunctions of the radial part of the Laplace–Beltrami operator on SS X:

$$\frac{1}{\sqrt{g}} \sum_{j=1}^{p} \frac{\partial}{\partial \alpha^j} \sqrt{g} \frac{\partial}{\partial \alpha^j} \Phi(\alpha) = (\sigma_1(\sigma_1 + p + q - 2) + \cdots + \sigma_p(\sigma_p + q - p))\Phi(\alpha). \qquad (3.26)$$

Transformation $\Phi(\alpha) = \sqrt[4]{g}\Psi(\alpha)$ reduces the equation (3.26) to the one-dimensional Schrödinger equation of the p-body system with potential

$$V = \sum_{i<j} \left(\frac{g_{\alpha_i - \alpha_j}}{\sinh^2(\alpha_i - \alpha_j)} + \frac{g_{\alpha_i + \alpha_j}}{\sinh^2(\alpha_i + \alpha_j)} + \frac{g_{\alpha_1}}{\sinh^2 \alpha_i} \right)$$
$$\text{where } g_{\alpha_i - \alpha_j}, g_{\alpha_i + \alpha_j}, g_{\alpha_i} > 0. \qquad (3.27)$$

Therefore the quantum systems related with the Laplace–Beltrami operator on SS X have only scattering states: hence plane waves are complete and we have orthogonal basis functions on SS X with the following completeness and orthogonality conditions:

$$\int \exp\left[\sum_{j=1}^{p} \sigma_j' \ln\{[x, s_j'] - [x, s_{j-1}'] \operatorname{tr}(n_{j-1})\}\right] \exp\left[\sum_{j=1}^{p} \sigma_j' \ln\{[x', s_j'] - [x', s_{j-1}] \operatorname{tr}(n_{j-1})\}\right]$$
$$\times \prod \mathrm{d}s_i \frac{\prod_{j=1}^{p} \mathrm{d}\rho_j}{|c(\sigma_1, \ldots, \sigma_p)|^2} = \delta(x' - x)$$
$$\int \exp\left[\sum_{j=1}^{p} \sigma_j' \ln\{[x, s_j'] - [x, s_{j-1}'] \operatorname{tr}(n_{j-1})\}\right] \exp\left[\sum_{j=1}^{p} \bar{\sigma}_j' \ln\{[x, \bar{s}_j'] - [x, \bar{s}_{j-1}'] \operatorname{tr}(n_{j-1})\}\right] \mathrm{d}x$$
$$= |c(\sigma_1, \ldots, \sigma_p)|^2 \prod \delta(\rho_j - \bar{\rho}_j)\delta(s_j - \bar{s}_j). \qquad (3.28)$$

Harish–Chandra's $c(\sigma_1, \ldots, \sigma_p)$-functions are represented in the form

$$c(\sigma_1, \ldots, \sigma_p) = \prod_{i=1}^{p} c_{p-i+1,p-i+1}(\sigma_i) \qquad (3.29)$$

where the expression of the $c_{p-i+1,q-i+1}$-functions ($k = 1, \ldots, p$) are defined by the formula (2.57) setting $p \to p - k + 1$, $q \to q - k + 1$. Indeed, the asymptotic expression of the zonal spherical functions have the form

$$t^{\chi}_{0_k;0_k}(g) \approx c_{p,q} \exp\left[\left(-\frac{p+q-2}{2} + \mathrm{i}\rho_1\right)\alpha^1\right] + \bar{c}_{p,q} \exp\left[\left(-\frac{p+q-2}{2} - \mathrm{i}\rho_1\right)\alpha^1\right]$$

when α^1 tends to infinity,

$$t^{\chi}_{0_k;0_k}(g) \approx c_{p-1,q-1} \exp\left[\left(-\frac{p+q-4}{2} + \mathrm{i}\rho_2\right)\alpha^2\right]$$
$$+ \bar{c}_{p-1,q-1} \exp\left[\left(-\frac{p+q-4}{2} - \mathrm{i}\rho_1\right)\alpha^2\right] \qquad (3.30)$$

when α^2 tends to infinity and so on.

Harish–Chandra's c-functions for complex semisimple Lie groups G were calculated by Gindikin and Karpelevich [15]. As shown by Helgason the expression for the c-function in the general case of a semisimple group G can be represented as product of the c-functions of the group G with real rank 1 [16].

4. Green functions

It follows from the results of the previous section that the integrable quantum systems considered above are related to the quantum free motion on SS. Namely, one-dimensional n-body integrable systems are a result of the broken symmetry of the quantum free motion on d-dimensional homogenous spaces. There exist many coordinate systems which reduce to the separation of variables in the Laplace–Beltrami operator [17] but only for those that are geodesics, which are related to one-parameter subgroups of the symmetry group, do there exist simple transformations of the Laplace–Beltrami operator on SS to some Hamiltonians of quantum systems. The quantum system depends on the way the symmetry is broken (see [5]).

The arbitrary quantum motion on SS was considered in a number of works. An exhaustive bibliography connected with this problem is presented in [6], where the quantum dynamics on compact Lie groups and homogenous spaces is considered.

First we consider the decomposition of the quasi-regular representation

$$T(g)F(x) = F(xg) \qquad g \in SO(1,q) \tag{4.1}$$

related to the hyperboloid $[x,x]=1$, $x_0>0$. This representation is unitary with respect to the scalar product:

$$\langle F_1, F_2\rangle = \int \overline{F_1}(x)F_2(x)\,\mathrm{d}x \tag{4.2}$$

where dx is an invariant volume on the hyperboloid. We define the Fourier components of the functions $F(x)$:

$$A(s,\rho) = \int F(x)[x,s]^{-(q-1)/2+\mathrm{i}\rho}\,\mathrm{d}x. \tag{4.3}$$

For the functions $F_g(x) = F(xg)$ we have

$$\begin{aligned} A(s,\rho) &= \int F_g(x)[x,s]^{-(q-1)/2+\mathrm{i}\rho}\,\mathrm{d}x = \int F(x)[xg^{-1},s]^{-(q-1)/2+\mathrm{i}\rho}\,\mathrm{d}x \\ &= (sg)_0^{-(q-1)/2+\mathrm{i}\rho}A(\overline{sg},\rho). \end{aligned} \tag{4.4}$$

Here we have used the relation

$$[xg^{-1},s]^\sigma = (sg)_0^\sigma[x,\overline{sg}]^\sigma \tag{4.5}$$

and the invariance of the volume element $\mathrm{d}x = \mathrm{d}(xg)$. Therefore the representation in the space of the functions $A(s,\rho)$ is irreducible. Decompositions of the representation equation (4.1) reduce to the Fourier expansion of the functions $F(x)$. Using the orthogonality condition for the plane waves equation (2.19) we have

$$F(x) = \frac{1}{(2\pi)^2}\int_0^\infty\int_{s^{q-1}} A(s,\rho)[x,s]^{-(q-1)/2\rho}\,\mathrm{d}s\,\frac{\mathrm{d}\rho}{|c(\rho)|^2}. \tag{4.6}$$

The Green functions of the quantum system with Hamiltonian $H = -(1/2mr^2)\Delta_{\mathrm{LB}}$ satisfies the equation

$$\left(-\frac{1}{2mr^2}\Delta_{\mathrm{LB}} - E\right)G(x_1,x_2) = \delta(x_1-x_2). \tag{4.7}$$

As is seen from the completeness conditions for plane waves the Green functions on SS $X \equiv SO(1,q)/SO(q)$ can be expressed in the form

$$G(x_1, x_2) = \frac{1}{(2\pi)^q} \int_0^\infty \int_{S^{q-1}} \frac{[x_1, s]^{-(q-1)/2+\mathrm{i}\rho}[x_2, s]^{-(q-1)/2-\mathrm{i}\rho}\,\mathrm{d}s}{[((q-1)/2)^2+\rho^2]/2mr^2 - E} \frac{\mathrm{d}\rho}{|c(\rho)|^2}. \tag{4.8}$$

Having used the relations given by equation (4.5) it is easy to get the invariant condition

$$G(x_1 g, x_2 g) = G(x_1, x_2). \tag{4.9}$$

Hence we have

$$G(x, \mathring{x}) = \frac{1}{(2\pi)^q} \int_0^\infty \int_{s^{q-1}} \frac{[x, s]^{-(q-1)/2-\mathrm{i}\rho}\,\mathrm{d}s}{[((q-1)/2)^2+\rho^2]/2mr^2 - E^2} \frac{\mathrm{d}\rho}{|c(\rho)|^2} \tag{4.10}$$

where $\mathring{x} = (1, 0, \ldots, 0)$ is the fixed point. This representation for the Green functions also follows from the addition theorem for the plane waves equation (2.20). So, the calculation of the Green functions reduces to the calculation of the zonal spherical functions. Using the known results [8]

$$\int_{S^{q-1}} [x, s]^{-(q-1)/2+\mathrm{i}\rho}\,\mathrm{d}s = \frac{2^{(q-2)/2}\Gamma(q/2)}{\sinh^{(q-2)/2}\alpha} P^{(2-q)/2}_{-1/2-\mathrm{i}\rho}(\cosh\alpha) \tag{4.11}$$

where $\cosh\alpha = [x, \mathring{x}]$ and $P^{\nu}_{\sigma}(z)$ is an associated Legendre function of the first kind, we get

$$G(x, \mathring{x}) = \frac{1}{2\pi} \int_0^\infty \frac{(2\pi\sinh\alpha)^{(2-q)/2}\,\mathrm{d}\rho}{[((q-1)/2)^2+\rho^2]/2mr^2 - E} \times \left|\frac{\Gamma(\mathrm{i}\rho+(q-1)/2)}{\Gamma(\mathrm{i}\rho)}\right|^2 P^{(2-q)/2}_{-1/2-\mathrm{i}\rho}(\cosh\alpha). \tag{4.12}$$

From this the energy-dependent Green function can be obtained in closed form:

$$G([x, \mathring{x}]) = \frac{m\mathrm{e}^{2\pi\mathrm{i}\varepsilon}}{\pi} \left(\frac{-1}{2\pi r^2\sinh\alpha}\right)^{(q-2)/2} Q^{(q-2)/2}_{-1/2-\mathrm{i}\nu}(\cosh\alpha) \tag{4.13}$$

where $\varepsilon = 0(1/2)$ for q odd (even) and $\nu = \sqrt{2mr^2E - ((d-1)/2)^2}$, Q^{μ}_{ν} is a Legendre function of the second kind (see appendix B). This result has been obtained in [18–20].

For the Green functions on the SS $X\ SO(p,q)/SO(p)\times SO(q)$ we have the following integral representation:

$$G([x, \mathring{x}]) = \int \frac{t^{\chi_1}_{0_k;0_k}(g_x)|\prod_{k=1}^{p} c_{p-k+1,q-k+1}|^{-2}}{(\sigma_1(\sigma_1+p+q-2)+\cdots+\sigma_p(\sigma_p+q-p))/2mr^2 - E} \prod_{j=1}^{p} \mathrm{d}\rho_j. \tag{4.14}$$

Thus calculation of the Green functions in the general case reduces to the calculation of the zonal spherical functions $t^{\chi_1}_{0_k;0_k}(g_x)$. But this problem has been open up to now. The calculation of the zonal spherical functions for the group $SU(p,q)$, $p \leqslant q$, has been performed by Berezin and Karpelevich [21]. The plane waves on SS $X \equiv SU(p,q)/S(U)(p)\times U(q))$ and integrable quantum systems related to this SS X will be considered in a subsequent publication.

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Appendix A. The maximal degenerate representations of the group $SO(p, q)$

Here we give the scalar products in the invariant subspaces of the maximal degenerate irreducible unitary representations of the group $SO(p, q)$.

We realise the representation (2.44) in the space D_χ, $\chi = (\sigma, \varepsilon)$, of the infinitely differentiable functions $f(s)$, $s = (\xi^p, \xi^q)$, $\xi^p \in S^{p-1}$, $\xi^q \in S^{q-1}$, on the sphere $S^{p-1} \times S^{q-1}$ with given parity $\varepsilon : f(s) = (-1)^\varepsilon f(s)$. We have

$$T_\chi(g) f(s) = \mathrm{e}^{\sigma(\alpha_g - \alpha)} f(\overline{sg}) \tag{A.1}$$

with respect to the scalar product

$$(f_1, f_2)_\chi = \int_{S^{p-1}} \int_{S^{q-1}} \overline{f_1(s)} f_2(s)\, \mathrm{d}s \tag{A.2}$$

where $\mathrm{d}s$ is an invariant volume on the sphere $S^{p-1} \times S^{q-1}$ and $\sigma = -(p+q-2)/2 + \mathrm{i}\rho$, $0 \leqslant \rho < \infty$ (principal series). It follows from that relation

$$\mathrm{d}s = \mathrm{e}^{\sigma(\alpha_g - \alpha)(p+q-2)}\, \mathrm{d}s'. \tag{A.3}$$

The Hermitian functional

$$(f_1, f_2)_\chi = c \int_{S^{p-1}} \int_{S^{q-1}} |[s^{(1)}, s^{(2)}]|^{-\sigma-p-q+2} \overline{f_1(s^{(1)})} f_2(s^{(2)})\, \mathrm{d}s^{(1)}\, \mathrm{d}s^{(2)} \tag{A.4}$$

is also invariant under representation (A.1).

In order to investigate positive definitions of $(f_1, f_2)_\chi$ it is convenient to represent it in canonical form. For this the calculation of the following integral is required:

$$\gamma_{l_p,l_q} = \frac{1}{b} \int_0^\pi \int_0^\pi |\cos\theta - \cos\omega|^{-\sigma-p-q+2} C_{l_p}^{(p-2)/p}(\cos\theta) C_{l_q}^{(q-2)/q}(\cos\omega) \times \sin^{p-2}\theta \sin^{q-2}\omega\, \mathrm{d}\theta\, \mathrm{d}\omega \tag{A.5}$$

where

$$b = \sqrt{\frac{\Gamma(p+l_p-2)\Gamma(q+l_q-2)\Gamma(p-1)\Gamma(q-1)}{l_p! l_q! (2l_p+p-2)(2l_q+q-2)}} \frac{2^{-p-q-8}\pi}{\Gamma^2((p-2)/2)\Gamma^2((q-2)/2)}$$

and $C_l^p(\cos\theta)$ are the Gegenbauer polynomials. From [7] we have

$$\begin{aligned}\beta_{l_p,l_q} &= \gamma^\chi_{l_p,l_q} \sqrt{\frac{l_p!\Gamma(p-1) l_q! \Gamma(q-1)}{\Gamma(l_p+p-2)\Gamma(l_q+q-2)(2l_p+p-2)(2l_q+q-2)}} \\ &= \{2^{\sigma+p+q-4}(-1)^{l_q} \tfrac{1}{2}[1+(-1)^{\varepsilon+l_p-l_q}]\Gamma((p-2)/2)\Gamma((q-2)/2) \\ &\quad \times\Gamma(-\sigma-p-q+3)\Gamma(-\sigma-((p+q)/2)+1)\}\{\Gamma((-\sigma/2) \\ &\quad +((l_p+l_q)/2))\Gamma((-\sigma-q+2)/2)+((l_p-l_q)/2))\Gamma((-\sigma-p-q+4)/2) \\ &\quad -((l_p+l_q)/2))\Gamma((-\sigma-p+2)/2)-((l_p-l_q)/2))\}^{-1}.\end{aligned} \tag{A.6}$$

It follows from (A.6) that the condition $(f_1, f_2)_\chi > 0$ or $\gamma^\chi_{l_p,l_q} > 0$ is fulfilled in the following cases:

(1)

(a) $(p+q)/2 < \sigma\langle(-p-q+4)/2$ when $p+q$ even, $\varepsilon = (p-q)/2$;

(b) $(1-p-q)/2 < \sigma < (-p-q+3)/2$ when $p+q$ odd, $\varepsilon = 0, 1$.

This is the complementary series. In this case we have

$$(f_1, f_2)_\chi = \sum_{l_p,l_q=0}^{\infty} \frac{\Gamma(((\sigma+p+q-2)/2)+((l_p+l_q)/2))\Gamma(((\sigma+p)/2)+((l_p-l_q)/2))}{\Gamma((-\sigma/2)+((l_p+l_q)/2))\Gamma(((-\sigma-q+2)/2)+((l_p-l_q)/2))} \times \sum_{K,M} \overline{a_{KM}^{(1)l_p,l_q}} a_{KM}^{(2)l_p,l_q}$$

where $a_{KM}^{(i)l_p,l_q}$ are generalized Fourier components of the functions $f^i(s)$ on the sphere $S^{p-1} \times S^{q-1}$.

(2) $-\sigma - p - q + 2 = L > -(p+q-2)/2$

(a) $l_p > l_q$, $l_p - l_q = L + q + 2n$, $n = 0, 1, 2, \ldots$, if $(-1)^{L+q} = (-1)^{\varepsilon}$;

(b) $l_q > l_q$, $l_q - l_p = L + p + 2n$, $n = 0, 1, 2, \ldots$, if $(-1)^{L+q} = (-1)^{\varepsilon}$.

These are discrete series. In these cases we have

(a)

$$(f_1, f_2)_\chi = \sum_{l_p=L+q}^{\infty} \sum_{l_q=0}^{l_p-L-q} [\Gamma((-L/2)+((l_p+l_q)/2))\Gamma(((-L-q+2)/2)+((l_p-l_q)/2))] \times [\Gamma(((L+p+q-2)/2)+((l_p+l_q)/2))\Gamma(((L+p/2)+((l_p-l_q)/2))]^{-1} \times \sum_{K,M} \overline{a_{KM}^{(1)l_p,l_q}} a_{KM}^{(2)l_p,l_q}$$

(b)

$$(f_1, f_2)_\chi = \sum_{l_q=L+p}^{\infty} \sum_{l_p=0}^{l_q-L-p} [\Gamma((-L/2)+((l_p+l_q)/2))\Gamma(((-L-p+2)/2)+((l_q-l_p)/2))] \times [\Gamma(((L+p+q-2)/2)+((l_p+l_q)/2))\Gamma(((L+p)/2)+((l_q-l_p)/2))]^{-1} \times \sum_{K,M} \overline{a_{KM}^{(1)l_p,l_q}} a_{KM}^{(2)l_p,l_q}.$$

Appendix B.

Here we give some formulae which have been used in the calculation of Green functions on the q-dimensional hyperboloid $[x, x] = 1$, $x_0 > 0$. For even $q = 2m$, m an integer, we have

$$\left| \frac{\Gamma(\mathrm{i}\rho + (q-1)/2)}{\Gamma(\mathrm{i}\rho)} \right|^2 P_{-1/2-\mathrm{i}\rho}^{(2-q)/2}(\cosh\alpha) = (-1)^{m-1} \rho \tanh \pi\rho P_{-1/2-\mathrm{i}\rho}^{m-1}(\cosh\alpha). \tag{B.1}$$

By the formula

$$P_\nu^m(z) = (z^2-1)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}z^m} P_\nu(z) \tag{B.2}$$

from (4.12) we get

$$\begin{aligned} G^{(1,q)}([x, \mathring{x}]) &= \frac{mr^2}{\pi} \left(-\frac{1}{2\pi} \frac{\mathrm{d}}{\mathrm{d}\cosh\alpha} \right)^{(d-2)/2} \times \int_0^\infty \frac{\rho \tanh \pi\rho \, \mathrm{d}\rho}{\rho^2 - [2mr^2 E - ((d-1)/2)^2]} P_{-1/2-\mathrm{i}\rho}(\cosh\alpha) \\ &= \frac{mr^2}{\pi} \left(-\frac{1}{2\pi} \frac{\mathrm{d}}{\mathrm{d}\cosh\alpha} \right)^{(d-2)/2} Q_{-1/2-\mathrm{i}\nu}(\cosh\alpha) \end{aligned} \tag{B.3}$$

where $\nu = \sqrt{2mr^2 E - ((d-1)/2)^2}$.

We have used the integral

$$\int_0^\infty \frac{x\tanh\pi x}{x^2+a^2} P_{-1/2-\mathrm{i}x}(\cosh\alpha)\,\mathrm{d}x = Q_{-1/2+a}(\cosh\alpha). \tag{B.4}$$

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